Lecture 15

Options hedging approaches

Agenda:

I. The factors affecting European stock option prices
   Greek letters for the Black-Scholes option pricing model
   1. Delta
   2. Gamma
   3. Vega
   4. Theta

II. Relation between Delta, Gamma, and Theta:

III. Hedging in practice
I. The factors affecting European stock option prices

I.1 Greek letters for the Black-Scholes option pricing model

$$\text{Option} = f(S, X, T, r, q, \sigma)$$

$$c = S_0 N(d_1) - Ke^{rT} N(d_2)$$

$$p = Ke^{rT} N(-d_2) - S_0 N(-d_1)$$

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma \sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}$$

**Delta:**

$$\frac{\partial \text{Option}}{\partial S} \sim \text{sensitivity of the option price change to a small change of } S$$

$$\frac{\partial \Pi}{\partial S} \sim \text{sensitivity of a portfolio’s value change to a small change of } S$$

**Gamma:**

$$\frac{\partial^2 \text{Option}}{\partial S^2} \sim \text{sensitivity of the delta change to a small change of } S$$

$$\frac{\partial^2 \Pi}{\partial S^2} \sim \text{sensitivity of a portfolio’s delta change to a change of } S$$

**Rho:**

$$\frac{\partial \text{Option}}{\partial r} \sim \text{sensitivity of the option price change to a small change of } r$$

**Vega:**

$$\frac{\partial \text{Option}}{\partial \sigma} \sim \text{sensitivity of the option price change to a small change of } \sigma$$

$$\frac{\partial \Pi}{\partial \sigma} \sim \text{sensitivity of a portfolio’s value change to a small change of } \sigma.$$  

**Theta (time decay):**

$$\frac{\partial \text{Option}}{\partial T} \sim \text{sensitivity of the option price change to the passage of time.}$$

$$\frac{\partial \Pi}{\partial T} \sim \text{sensitivity of a portfolio’s value change to the passage of time.}$$
\textbf{1.2. Delta}

\~ \textbf{Delta}: The sensitivity of option price change to a small stock price change

\[
\text{Call Delta} = \frac{\partial C}{\partial S} = N(d_1) \quad 0 \leq N(d_1) \leq 1
\]

\[
\text{Put Delta} = \frac{\partial P}{\partial S} = N(d_1) - 1 \quad -1 \leq N(d_1) - 1 \leq 0
\]

\~ \textbf{Delta hedging}: – option + delta \times S; this portfolio is called a Delta neutral portfolio.

Perfect delta hedging:

If S changes, we need to rebalance the hedging position continuously.

If S changes, delta will change, we need to rebalance the delta of the portfolio continuously.
~ Delta hedging:

\[ S_0 = \$100 \]
\[ C = \$10 \]

- Short 1 call: Buy 100 \( \times \) Delta = 60 shares
  - \( \Delta C = +\Delta S \times \Delta \)
  - if \( \Delta S = +$1 \) (from $100 to $101)

The change of call price: \( $1 \times 0.6 \times 100 \text{ shares} = $60 \)
The change of stock position: \( $1 \times 60 \text{ shares} = $60 \)

- Dynamic hedging v.s. Static-hedging:
  As stock price keeps changing, the delta will change. Thus, we need to rebalance the portfolio in order to maintain the delta neutral condition.

\( S \rightarrow $110, \Delta \rightarrow 0.65. \)
We need to add extra 5 shares of stock into the portfolio. It’s called dynamic-hedging. If we just leave it alone, it’s called static-hedging.

~ Delta hedging v.s Stop-loss hedging:

Stop-loss hedging for a short call: as \( S > K \), buy a stock at \( K \).
as \( S < K \), sell a stock at \( K \).

~ Delta hedging based on Futures contracts:

\[ F = S_0 e^{(r-q)T} \]
\[ \Delta = e^{(r-q)T} \]
Number of Futures contract in hedging = \( e^{-(r-q)T} \times \Delta \text{ of the option} \)
\( (\Delta + N \times e^{-(r-q)T} = 0 \rightarrow N = \Delta / e^{-(r-q)T}) \).

~ Delta of a portfolio:

\[ \Delta_{\text{port}} = \sum_{i=1}^{n} w_i \Delta_i \]
~ Problem of Delta-neutral hedging:

If Delta is extremely sensitive to stock price changes, we need to rebalance the portfolio continuously.

![Graph showing the relationship between call price and stock price with notation for estimation error.]

- The estimation error is determined by the curvature of the relation between call and stock prices. A big change of S will have big estimation error.
I. 3 Gamma

- **Gamma**: Sensitivity of the delta change to a small change of $S$

$$\text{Call Gamma} = \text{Put Gamma} = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}} = \frac{e^{-d_1^2/2}}{S_0 \sigma \sqrt{2\pi T}} > 0$$

If Gamma is small (big), delta changes slowly (quickly).

* The delta of ATM options have the highest sensitivity to a stock price change.

* For ATM options, as time passes away, the gamma increases dramatically, because ATM value is very sensitive to jumps in stock prices.
~ Gamma and portfolio’s value:

\[
\Gamma = \frac{\partial^2 \Pi}{\partial S^2}, \quad \Gamma_{\text{Port}} = \sum_{i=1}^{n} w_i \Gamma_i
\]

\[
\delta \Pi = \frac{\partial \Pi}{\partial S} \delta S + \frac{\partial \Pi}{\partial t} \delta t + \frac{1}{2} \frac{\partial^2 \Pi}{\partial S^2} (\delta S)^2 + …
\]

If \( \Gamma_{\text{Port}} > 0 \), the value of the portfolio will increase as \( S \) moves (either up or down).
If \( \Gamma_{\text{Port}} < 0 \), the value of the portfolio will decrease as \( S \) moves (either up or down).

~ Delta-gamma-hedging:

The Gamma of a Delta-neutral portfolio (-C + Delta \times S): \( \Gamma \)

To make the delta-neutral portfolio into a Delta-gamma neutral portfolio, we need to

1) add certain amount of other options into the portfolio:

\[
N_G \times \Gamma_G + \Gamma = 0 \quad (N_G \text{ is number of new options; } \Gamma_G \text{ is gamma of the new options})
\]

\[
N_G = -\frac{\Gamma}{\Gamma_G}
\]

2) adjust number of stocks to make the new portfolio delta-neutral:

Example: A Delta-neutral portfolio:
shorts 1 Call with a Delta of 0.6 and gamma of 1.5
longs 60 shares of stock

Delta of the portfolio: \(-0.6 \times 100 + 1 \times 60 = 0\)
Gamma of the portfolio: \(-1.5 \times 100 + 0 \times 60 = -150\)

If we would like to use another call options with a Delta of 0.5 and gamma of 2 to construct a Delta-gamma-neutral portfolio:

1) \[
N_G = -\frac{\Gamma}{\Gamma_G} = -\frac{-150}{200} = 0.75
\] Long 0.75 of the new option
\[
\Gamma = -150 + 0.75 \times 200 = 0
\]

2) Delta of the new portfolio: \(0.75 \times 100 \times 0.5 = 37.5\)
Sell 37.5 shares of the stock.

The Delta-gamma-neutral portfolio:
Short one call with Delta of 0.6 and gamma of 1.5
Long 0.75 call with Delta of 0.5 and gamma of 2
Long 22.5 (60-37.5) shares of stock.
I. 5 Vega:

Vega: sensitivity of the option price change to a small change of $\sigma$

Call $\text{vega} = \text{Put vega} = S_0 \sqrt{T} \text{ } N'(d_1) > 0$

- ATM options have the highest sensitivity to volatility change. The volatility and ATM options are almost linearly related.
~ Delta-Gamma-Vega hedging:

A Delta-neutral portfolio:
shorts 1 Call (Delta=0.6, Gamma=1.5, Vega=1.2)
longs 60 shares of stock (Delta=1, Gamma=0, Vega=0)

Delta of the portfolio: $-0.6 \times 100 + 1 \times 60 = 0$
Gamma of the portfolio: $-1.5 \times 100 + 0 \times 60 = -150$
Vega of the portfolio: $-1.2 \times 100 = -120$

To construct a Delta-gamma-neutral portfolio, we need a new option.

To construct a Delta-gamma-vega-neutral portfolio, we need another new option.

C₁: Delta=0.5, Gamma=2, Vega=1.5
C₂: Delta=0.4, Gamma=0.8, Vega=1

1) $-150 + 2 \times N₁ + 0.8 \times N₂ = 0$ (make the portfolio Gamma-neutral)
   $-120 + 1.5 \times N₁ + 1 \times N₂ = 0$ (make the portfolio Vega-neutral)

   \[
   \begin{align*}
   N₁ &= 67.5 \quad \text{we need } 67.5/100 = 0.675 \text{ C₁} \\
   N₂ &= 18.75 \quad \text{we need } 18.75/100 = 0.1875 \text{ C₂}
   \end{align*}
   \]

2) Delta of the new portfolio $= 0.675 \times 100 \times 0.5 + 0.1875 \times 100 \times 0.4 = 41.25$
We need to short 41.25 shares of the stock to make the portfolio Delta-neutral.

The Delta-gamma-vega-neutral portfolio:
Short one call with Delta of 0.6, gamma of 1.5, and vega of 1.2.
Long 0.675 call with Delta of 0.5 and gamma of 2, and vega of 1.5
Long 0.1875 call with Delta of 0.4 and gamma of 0.8, and vega of 1
Long 18.75 (60-41.25) shares of stock.
I. 6 Theta (time decay):

Theta: sensitivity of the option price change to the passage of time.

Call Theta = $- \frac{S_0 N'(d_1) \sigma}{2 \sqrt{T}} - rKe^{-rT} N(d_1)$  Usually ≤ 0

Put Theta = $- \frac{S_0 N'(d_1) \sigma}{2 \sqrt{T}} + rKe^{-rT} N(-d_2)$  Usually ≤ 0

• We don’t need to hedge Theta risk. There is no uncertainty about the passage of time.

• $S_0 \rightarrow 0$, Theta $\rightarrow 0$
  $S_0 \rightarrow \infty$, Theta $\rightarrow -rKe^{-rT}$

|ATM Theta| > |ITM Theta| > |OTM Theta|

*Intuitively, it is because ATM time value > ITM time value > OTM time value.*

• As time passes away, options are more and more sensitive to the passage of time. However, as time to maturity approaches to 0, options become less sensitive to the passage of time.
II. Relation between Delta, Gamma, and Theta

According to the Black-Sholes-Merton differential equation, a derivative must satisfy:

$$\frac{\partial f}{\partial t} + r \frac{\partial f}{\partial S} S + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 = rf$$

The value of a portfolio of such derivatives, $\Pi$, shall also satisfy:

$$\frac{\partial \Pi}{\partial t} + r \frac{\partial \Pi}{\partial S} S + \frac{1}{2} \frac{\partial^2 \Pi}{\partial S^2} \sigma^2 S^2 = r \Pi$$

Theta = $\frac{\partial \Pi}{\partial t}$

Delta = $\frac{\partial \Pi}{\partial S}$

Gamma = $\frac{\partial^2 \Pi}{\partial S^2}$

Theta + $r \times$ Delta + $\frac{1}{2}$ Gamma × $\sigma^2 \times S^2$ = $r \Pi$

In a Delta-neutral portfolio, Delta=0,

$$\text{Theta} + \frac{1}{2} \Gamma \sigma^2 S^2 = r \Pi$$

• Given $r \Pi$, it indicates that a portfolio with a large and positive Gamma must have a large and negative Theta.

~ A large and positive Gamma is associated with a large a negative Theta.
~ A portfolio is positively related to Gamma, must be negatively related to Theta.
~ Theta can be regarded as a proxy for Gamma in a Delta-neutral portfolio.
• Connection of the Greek letters:

If volatility is also uncertain:

$$\delta \Pi = \frac{\partial \Pi}{\partial S} \delta S + \frac{\partial \Pi}{\partial \sigma} \delta \sigma + \frac{\partial \Pi}{\partial t} \delta t + \frac{1}{2} \frac{\partial^2 \Pi}{\partial S^2} \delta S^2 + \frac{1}{2} \frac{\partial^2 \Pi}{\partial \sigma^2} \delta \sigma^2 + \frac{1}{2} \frac{\partial^2 \Pi}{\partial t^2} \delta t^2 + \frac{\partial^3 \Pi}{\partial S\partial \sigma} \delta S \delta \sigma + \frac{\partial^3 \Pi}{\partial S\partial t} \delta S \delta t + \frac{\partial^3 \Pi}{\partial \sigma\partial t} \delta \sigma \delta t + \ldots$$

$$\delta \Pi = \frac{\partial \Pi}{\partial S} \delta S + \frac{\partial \Pi}{\partial \sigma} \delta \sigma + \frac{\partial \Pi}{\partial t} \delta t + \frac{1}{2} \frac{\partial^3 \Pi}{\partial S^2} \delta S^2$$

Delta Vega Theta Gamma

~ Delta hedging is to eliminate the first term: \(\frac{\partial \Pi}{\partial S} \delta S = 0\)

~ Vega hedging is to eliminate the second term: \(\frac{\partial \Pi}{\partial \sigma} \delta \sigma = 0\)

~ Gamma hedging is to eliminate the fourth term: \(\frac{1}{2} \frac{\partial^3 \Pi}{\partial S^2} \delta S^2 = 0\)

~ Delta-Gamma-Vega hedging is to eliminate all the three terms above.

~ We don’t hedge the theta because the passage of time is certain.
III. Hedging in practice

~ Usually, traders conduct delta-hedging on daily basis.

Traders usually leave gamma and vega risks alone for three reasons:

1. It’s difficult and expensive to find another suitable option to conduct the gamma and vega hedging.

2. As time goes by, options tend to become OTM (DOTM), or ITM (DITM). Thus, gamma and vega will diminishing by themselves.

3. According to the sensitivity analysis: by changing the variables in option pricing models, traders monitor the behaviors of gamma and vega continuously.
~ Hedging based on futures contracts:

• To hedge the risk of a stock, we can buy put options.

Put Delta = \( N(d_1) - 1 \)

\[
d_1 = \frac{\ln \left( \frac{S_0}{K} \right) + (r + \sigma^2/2)T}{\sigma \sqrt{T}}
\]

\[= \frac{\ln \left( \frac{100}{90} \right) + (0.05 + \frac{0.25^2}{2})0.5}{0.25 \sqrt{0.5}} \approx -0.204
\]

• Example: +S +Put \( (S_0=100, K=90, r=5\%, T=0.5, \sigma=25\%, q=0) \)

Delta of the put = -0.204

It means we need to short a derivative with a delta of -0.204.

If we would like to use futures to create a -0.204 delta, how many futures contract should we purchase?

Delta of a Futures = \( e^{(r-q)T} = e^{0.05 \times 0.5} = 1.0253 \)

Delta of the put = -0.204 = Delta of futures \( \times N_F \)

\[N_F = \frac{-0.204}{1.0253} = -0.199
\]

Thus, to hedge 1 share of the stock, we need to short 0.199 contracts of futures.

As Stock price changes, we need to rebalance the futures positions continuously.

~ If S drops to $80, the delta of the put becomes -0.669. The futures contracts needed in the hedging is:

\[N_F = \frac{-0.669}{1.0253} = -0.652
\]

Thus, to hedge 1 share of the stock, we need to short another 0.453 (0.652-0.199) contracts of futures.

~ If S jumps to $120, the delta of the put becomes -0.0316. The futures contracts needed in the hedging is:

\[N_F = \frac{-0.0316}{1.0253} = -0.0308
\]

Thus, to hedge 1 share of the stock, we need to buy back 0.168 (0.199-0.0308) contracts of futures.

• Pros and cons of hedging based on futures contracts:

Pros: hedging cost is low.

Cons: rebalance the futures positions continuously. ineffective if price drops dramatically.